SOLUTIONS OF PROBLEMS.

489 (Algebra). Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a\begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b\begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab\begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2 .$$

II. SOLUTION BY A. M. HARDING, University of Arkansas.

The quantities X, Y, U, V, W, defined by the equations

$$\begin{aligned} xX - ayY - buU + abvV &= -xW, \\ yX + xY - bvU - buV &= yW, \\ uX + avY + xU + ayV &= uW, \\ vX - uY + yU - xV &= -vW, \end{aligned}$$

are proportional to the five determinants taken in order.

It can be easily shown that the last determinant in the left member of the proposed equation is equal to zero for all values of x, y, u, v, a, b. Hence the above equations may be written in the form

$$\begin{aligned} (X + W)x - aYy - bUu &= 0, \\ Yx + (X - W)y - bUv &= 0, \\ Ux + (X - W)u + aYv &= 0, \\ Uy - Yu + (X + W)v &= 0. \end{aligned}$$

The quantities x, y, u, v, can satisfy this system of homogeneous linear equations if, and only if, the determinant

$$\begin{vmatrix} X + W & -aY & -bU & 0 \\ Y & X - W & 0 & -bU \\ U & 0 & X - W & aY \\ 0 & U & -Y & X + W \end{vmatrix}$$

is equal to zero. It can be shown that the value of this determinant is k^2, where

$$k = (X^2 - W^2) + aY^2 + bU^2.$$

Hence $X^2 + aY^2 + bU^2 = W^2$, or, since $V = 0$, $X^2 + aY^2 + bU^2 + abV^2 = W^2$. That is, the relation stated in the problem holds for all values of x, y, u, v, a, b.

NOTE. We are publishing a second solution of this problem for two reasons: First, because the solution above is essentially different from the one published in the March number; and second, because the conclusion drawn in that solution that the identity does not always exist is incorrect. EDITORS.

271 (Number Theory). Proposed by HORACE OLSON, Chicago, Illinois.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

Solution by the Proposer.

I shall first prove some lemmas.

1) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 3. This follows from the fact that any perfect square is congruent, modulo 3, to either 0 or 1.

2) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 4; for any perfect square is congruent, modulo 16, to 0, 1, 4, or 9.

3) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then at least one of the numbers x, y, u is livisible by 5; for any perfect square is congruent, modulo 5, to 0, 1, or 4.

Hence, from the hypotheses, at least two of the numbers x, y, z are divisible by 3. If the notation be so chosen that these are x and y, the first equation shows that u also is divisible by 3. We can then divide both terms of this equation by 9; it then follows that either $x/3$ or $y/3$ is divisible by 3. Thus the product $xyzuvw$ is divisible by 3^4. In the same way it can be proved that the product is divisible by 4^4.

By lemma 3, at least one of the numbers x, y, u is divisible by 5. Similarly, at least one of each of the sets x, z, v and y, z, w is divisible by 5. Thus at least two of the numbers x, y, z, u, v, w are divisible by 5.

Therefore the product $xyzuvw$ is divisible by $3^4 \cdot 4^4 \cdot 5^2$, or 518400.

Also solved by H. C. Feemster and J. L. Riley.

2661. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find a parallelepipedon whose edges, and the diagonals of its faces, are all rational whole numbers.

Solution by the Proposer.

Denote the edges by x, y, and z; then

$$x^2 + y^2 = \square, \quad x^2 + z^2 = \square, \quad y^2 + z^2 = \square. \tag{1, 2, 3}$$

Assume $x = 2pq$, $y = p^2 - q^2$; then by substitution,

$$x^2 + y^2 = (2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2,$$

a square;

$$x^2 + z^2 = (2pq)^2 + z^2, \quad y^2 + z^2 = (p^2 - q^2)^2 + z^2, \tag{4, 5}$$

which must be made squares.

Put

$$(2pq)^2 + z^2 = \left(z - \frac{2pqr}{s}\right)^2,$$

which gives

$$z = \frac{pq(r^2 - s^2)}{rs}.$$

Substituting in (5) the value of z found above,

$$(p^2 - q^2)^2 + \frac{p^2q^2(r^2 - s^2)^2}{r^2s^2} = \square,$$

or

$$r^2s^2(p^2 - q^2)^2 + p^2q^2(r^2 - s^2)^2 = \square. \tag{6}$$

Expanding (6) it may be written

$$(p^2r^2 + q^2s^2)(q^2r^2 + p^2s^2) - 4p^2q^2r^2s^2 = \square,$$

which will be the case when

$$q^2r^2 + p^2s^2 = 4q^2s^2. \tag{7}$$

By transposition, (7) becomes

$$q^2r^2 = 4q^2s^2 - p^2s^2 = s^2(4q^2 - p^2) = s^2\left(\frac{mp}{n} - 2q\right)^2,$$

say; and we get

$$\frac{p}{q} = \frac{4mn}{m^2 + n^2}, \qquad \frac{r}{s} = \frac{2(m^2 - n^2)}{m^2 + n^2};$$